ABICに基づくインバージョン解析手法の発展

Development of the method of inversion analyses using ABIC

Yukitoshi Fukahata (Kyoto University)

Bad modelling leads to a bad inversion result.
A bad inversion result suggests bad modeling.
悪いインバージョン結果には何か理由がある

Contents

0. Framework of inversion analysis using ABIC

1. More than one sort of prior information
   [Fukahata *et al*. 2004, 2003]

2. More than one sort of data (Joint inversion)
   [Fuuning, Fukahata, Yagi & Parsons 2014]

3. Weak non-linear inversion [Fukahata & Wright 2008]

4. Covariance components due to observation error
   [Fukahata & Wright 2008]

5. Covariance components due to modeling error
   -1. Discretization error [Yagi & Fukahata 2008]
   -2. Errors of Green's function [Yagi & Fukahata 2011]

Reference: 地震学におけるABICを用いたインバージョン解析研究の進展, 深畑, 地震２, S103-S113, 2009.

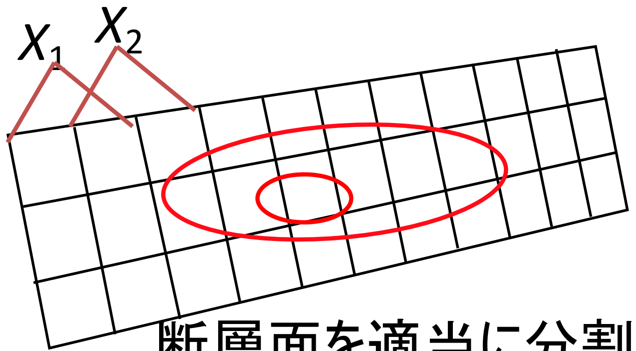
Inversion Algorithm Using ABIC for slip inversion (1)

(Based on Yabuki & Matsu'ura, 1992)

1. Parametric Expansion

$$u(x,y) = \sum_{k=1}^{m} \sum_{l=1}^{L} a_{kl} X_k(x) Y_l(y)$$

a_{kl}: model parameter, u: slip distribution,
$X_k(x)$, $Y_l(y)$: basis function



断層面を適当に分割して
滑り分布を基底関数展開

2. Observation Equation

$$\mathbf{d} = \mathbf{H}\mathbf{a} + \mathbf{e}$$

d: data, **e**: error, $\mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{E})$

$$p\left(\mathbf{d} \mid \mathbf{a}; \sigma^2\right) = (2\pi\sigma^2)^{-\frac{n}{2}} |\mathbf{E}|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(\mathbf{d}-\mathbf{H}\mathbf{a})^T \mathbf{E}^{-1}(\mathbf{d}-\mathbf{H}\mathbf{a})\right]$$

断層運動による地表の変形応答(e.g., Okada, 1985)が必要

3. Prior Information (smoothness condition)

$$\int_{XY} \left[(\partial u/\partial x)^2 + (\partial u/\partial y)^2\right] dxdy = \mathbf{a}^T \mathbf{G} \mathbf{a} \quad \rightarrow \quad \text{small}$$

$$p(\mathbf{a}; \rho^2) = (2\pi\rho^2)^{-\frac{m}{2}} |\mathbf{G}|^{\frac{1}{2}} \exp\left[-\frac{1}{2\rho^2} \mathbf{a}^T \mathbf{G} \mathbf{a}\right]$$

Inversion Algorithm Using ABIC for slip inversion (2)

(Based on Yabuki & Matsu'ura, 1992)

4. Bayes' Theorem（観測からの情報と先験的情報を統合）

$$p(\mathbf{a};\sigma^2,\rho^2 \mid \mathbf{d}) = c\,p(\mathbf{d} \mid \mathbf{a};\sigma^2)\,p(\mathbf{a};\rho^2)$$

$$= c\left(2\pi\sigma^2\right)^{-(n+m)/2}(\alpha^2)^{P/2}\left|\mathbf{E}\right|^{-1/2}\left|\mathbf{G}\right|^{1/2} \times \exp\left[-\frac{1}{2\sigma^2}s(\mathbf{a};\alpha^2)\right]$$

with

$$s(\mathbf{a};\alpha^2) = \left(\mathbf{d}-\mathbf{Ha}\right)^T \mathbf{E}^{-1}\left(\mathbf{d}-\mathbf{Ha}\right) + \alpha^2\mathbf{a}^T\mathbf{Ga} \qquad (\alpha^2 = \sigma^2/\rho^2 : \text{ hyperparameter})$$

5. ABIC Minimum (ABIC: Akaike's Bayesian Information Criterion)

$$\mathrm{ABIC}\left(\sigma^2,\alpha^2\right) = -2\log\int p\left(\mathbf{a};\sigma^2,\alpha^2 \mid \mathbf{d}\right)d\mathbf{a} + C$$

The criterion of ABIC minimum $\quad\rightarrow\quad \hat{\sigma}^2,\hat{\alpha}^2$

Given σ^2, α^2, based on maximum likelihood method

$$\rightarrow \quad \hat{\mathbf{a}} = \left[\mathbf{H}^T\mathbf{E}^{-1}\mathbf{H} + \hat{\alpha}^2\mathbf{G}\right]^{-1}\mathbf{H}^T\mathbf{E}^{-1}\mathbf{d}$$

$$\hat{\mathbf{C}} = \hat{\sigma}^2(\mathbf{H}^T\mathbf{E}^{-1}\mathbf{H} + \hat{\alpha}^2\mathbf{G})^{-1}$$

Observation eq.

$$\mathbf{d} = \mathbf{H}\mathbf{a} + \mathbf{e}; \quad p(\mathbf{d} \mid \mathbf{a}; \sigma^2)$$

Prior information

$$r = \mathbf{a}^T \mathbf{G} \mathbf{a}; \quad p(\mathbf{a}; \rho^2)$$

Bayes Thorem

$$p(\mathbf{a}; \sigma^2, \rho^2 \mid \mathbf{d}) = c\, p(\mathbf{d} \mid \mathbf{a}; \sigma^2)\, p(\mathbf{a}; \rho^2)$$

$$= c\left(2\pi\sigma^2\right)^{-(n+m)/2} (\alpha^2)^{P/2} \left|\mathbf{E}\right|^{-1/2} \left|\mathbf{G}\right|^{1/2} \times \exp\left[-\frac{1}{2\sigma^2} s(\mathbf{a}; \alpha^2)\right]$$

with $s(\mathbf{a}) = (\mathbf{d} - \mathbf{H}\mathbf{a})^T \mathbf{E}^{-1} (\mathbf{d} - \mathbf{H}\mathbf{a}) + \dfrac{\sigma^2}{\rho^2} \mathbf{a}^T \mathbf{G} \mathbf{a}$

$$\mathrm{ABIC}(\sigma^2, \rho^2) \equiv -2 \log\left[\int p(\mathbf{a}; \sigma^2, \rho^2 \mid \mathbf{d})\, d\mathbf{a}\right] + C$$

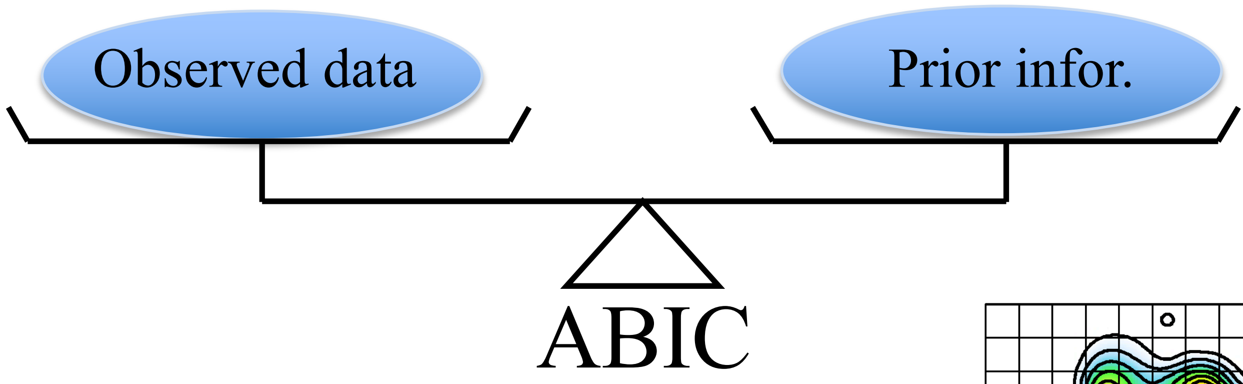
(Akaike 1980)

minimum ABIC ⇒ Optimal σ^2, ρ^2 (hyperparameter)

ABIC enables us to determine the relative weight between observed data and prior information objectively.

ABIC determines the relative weight based on statistics

Characteristic of ABIC

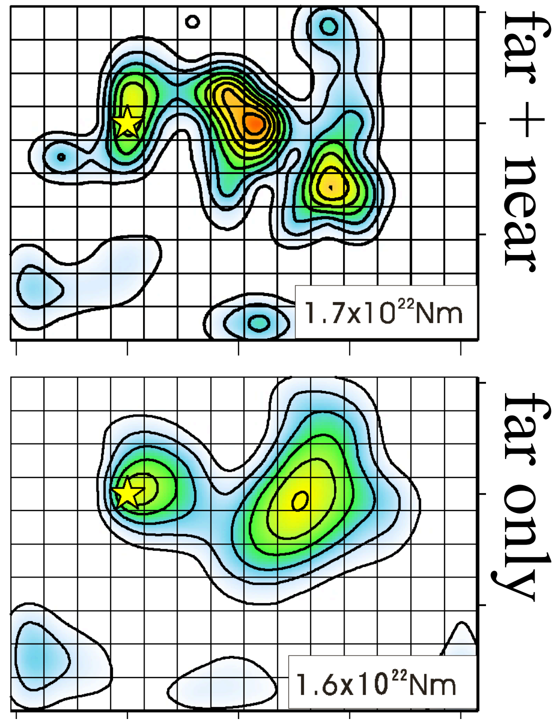


Observed data: accurate, sufficient

→ the model that fits to the data

Observed data: inaccurate, insufficient

→ the model that follows prior



Fukahata, Yagi, & Matsu'ura (2003)

Extension of inversion methods using ABIC

CASE1: **More than one sort of prior information**

Prior Information

$$\begin{cases} \int_T \int_X \left[\frac{\partial^2 u(x,t)}{\partial x^2} \right] dx\,dt = \mathbf{a}^T \mathbf{G}_1 \mathbf{a} \\ \int_T \int_X \left[\frac{\partial^2 u(x,t)}{\partial t^2} \right] dx\,dt = \mathbf{a}^T \mathbf{G}_2 \mathbf{a} \end{cases}$$

probability density function

$$\begin{cases} p(\mathbf{a};\rho_1^2) = (2\pi)^{-\frac{m}{2}} \left| \frac{\mathbf{G}_1}{\rho_1^2} \right|^{\frac{1}{2}} \exp\left[-\frac{1}{2\rho_1^2} \mathbf{a}^T \mathbf{G}_1 \mathbf{a} \right] \\ p(\mathbf{a};\rho_2^2) = (2\pi)^{-\frac{m}{2}} \left| \frac{\mathbf{G}_2}{\rho_2^2} \right|^{\frac{1}{2}} \exp\left[-\frac{1}{2\rho_2^2} \mathbf{a}^T \mathbf{G}_2 \mathbf{a} \right] \end{cases}$$

$$\Downarrow$$

$$p(\mathbf{a};\rho_1^2,\rho_2^2) = (2\pi)^{-m} \left| \frac{\mathbf{G}_1}{\rho_1^2} \right|^{\frac{1}{2}} \left| \frac{\mathbf{G}_2}{\rho_2^2} \right|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \mathbf{a}^T \left(\frac{\mathbf{G}_1}{\rho_1^2} + \frac{\mathbf{G}_2}{\rho_2^2} \right) \mathbf{a} \right]$$

(ex, Yoshida, 1989
Ide et al. 1996)

$$\int p(\mathbf{a};\rho_1^2,\rho_2^2)\, d\mathbf{a} \neq 1 \quad \Leftarrow \quad P(A \cap B) \neq P(A)P(B)$$

$$\Rightarrow \quad p(\mathbf{a};\rho_1^2,\rho_2^2) = (2\pi)^{-\frac{m}{2}} \left| \frac{\mathbf{G}_1}{\rho_1^2} + \frac{\mathbf{G}_2}{\rho_2^2} \right|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \mathbf{a}^T \left(\frac{\mathbf{G}_1}{\rho_1^2} + \frac{\mathbf{G}_2}{\rho_2^2} \right) \mathbf{a} \right]$$

規格化定数の重要性

(Fukahata et al., 2004, 2003)

Expression of ABIC

Proper

$$\mathrm{ABIC}(\alpha^2, \beta^2) = N \log s(\mathbf{a}^*) - \log \|\alpha^2 \mathbf{G}_1 + \beta^2 \mathbf{G}_2\| + \log \|\mathbf{H}^T\mathbf{H} + \alpha^2 \mathbf{G}_1 + \beta^2 \mathbf{G}_2\| + C'$$

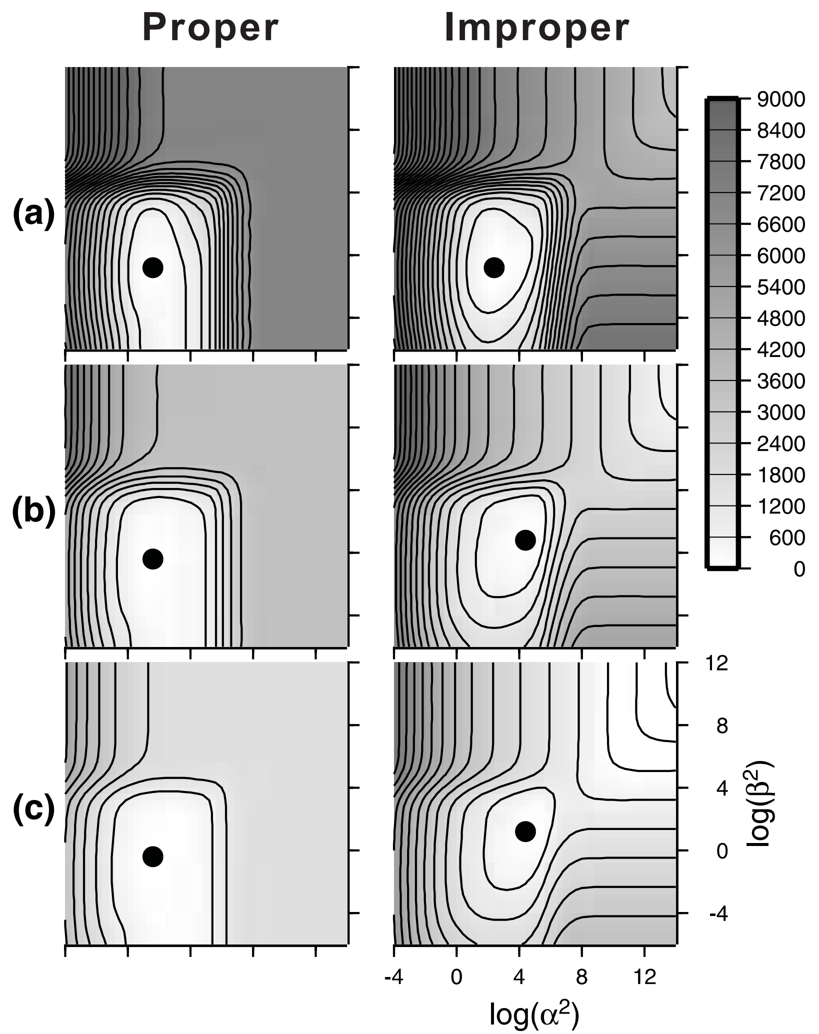
Improper

$$\mathrm{ABIC}(\alpha^2, \beta^2) = (N + P_1 + P_2 - M) \log s(\mathbf{a}^*) - \log \alpha^{2P_1}\beta^{2P_2} + \log \|\mathbf{H}^T\mathbf{H} + \alpha^2 \mathbf{G}_1 + \beta^2 \mathbf{G}_2\| + C''$$

(α → 大；　β → 大）　⇒　ABIC → マイナス無限大

$$\left(\alpha^2 = \frac{\sigma^2}{\rho_1^2},\ \beta^2 = \frac{\sigma^2}{\rho_2^2}\right)$$

ABICコンター



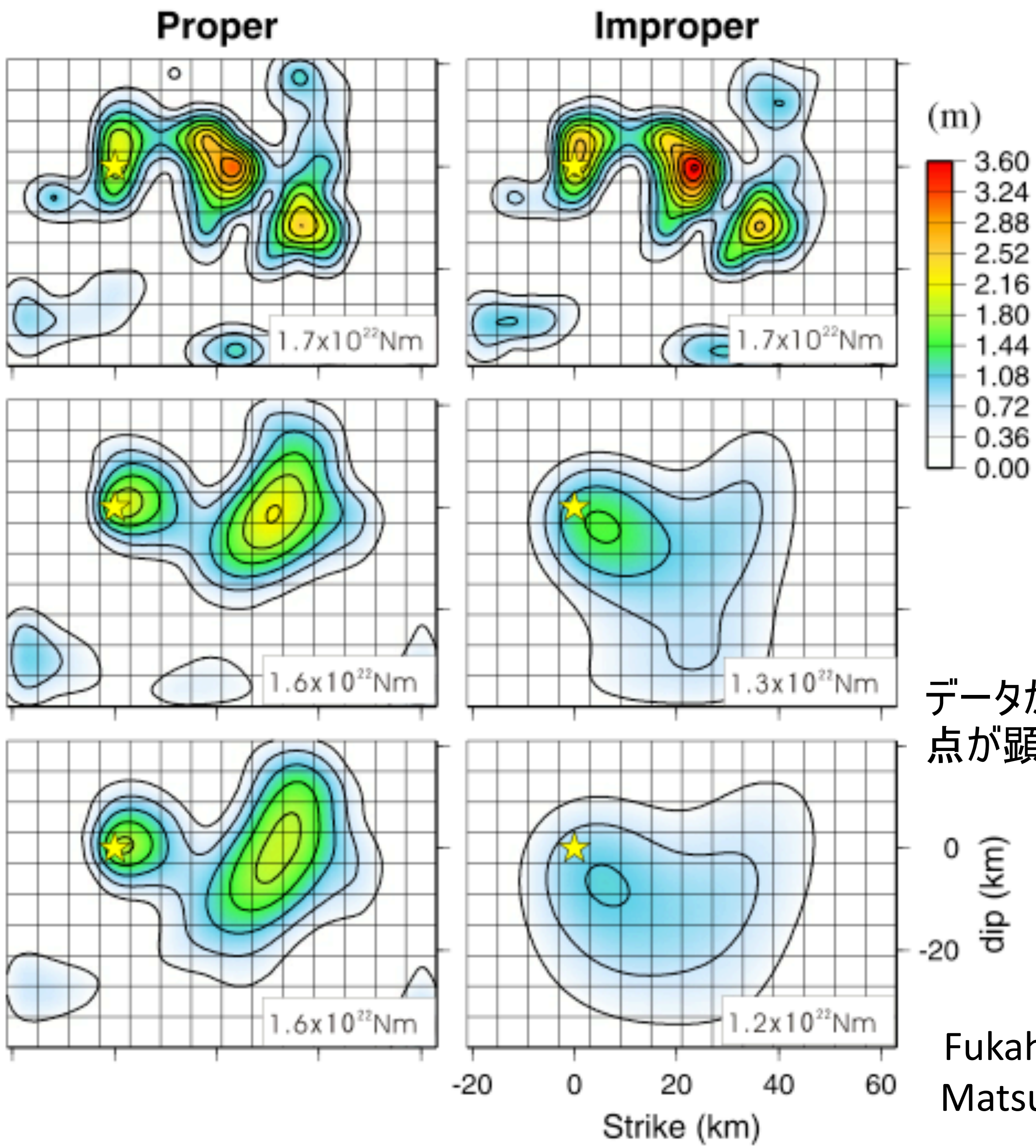
Fukahata, Yagi & Matsu'ura (2003)

Inverted Slip Distribution

Near(21) + Far(15)

Far(15)

Far(7)



データが少ないとき欠点が顕わに

Fukahata, Yagi & Matsu'ura (2003)

CASE2: **More than one sort of data (Joint inversion)**

Expression of observation equation

InSAR: $\mathbf{d}_I = \mathbf{H}_I\mathbf{a} + \mathbf{e}_I, \quad \mathbf{e}_I \sim N(0, \sigma_I^2\mathbf{E}_I)$

seismic: $\mathbf{d}_s = \mathbf{H}_s\mathbf{a} + \mathbf{e}_s, \quad \mathbf{e}_s \sim N(0, \sigma_s^2\mathbf{E}_s)$

$$p\left(\mathbf{d}_s \mid \mathbf{a};\sigma_s^2\right) = (2\pi\sigma_s^2)^{-\frac{N_s}{2}} \left|\mathbf{E}_s\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma_s^2}\left(\mathbf{d}_s - \mathbf{H}_s\mathbf{a}\right)^T \mathbf{E}_s^{-1}\left(\mathbf{d}_s - \mathbf{H}_s\mathbf{a}\right)\right]$$

⇒ *joint*

$$\mathbf{d} = \mathbf{H}\mathbf{a} + \mathbf{e}, \quad \mathbf{e}_I \sim N(0, \sigma_I^2\mathbf{E})$$

$$\begin{pmatrix}\mathbf{d}_I \\ \mathbf{d}_s\end{pmatrix} = \begin{pmatrix}\mathbf{H}_I \\ \mathbf{H}_s\end{pmatrix}\mathbf{a} + \begin{pmatrix}\mathbf{e}_I \\ \mathbf{e}_s\end{pmatrix} \quad \sigma_I^2\mathbf{E} = \begin{pmatrix}\mathbf{E}_I & 0 \\ 0 & \eta^2\mathbf{E}_s\end{pmatrix}$$

$$p\left(\mathbf{d} \mid \mathbf{a};\sigma_I^2,\eta^2\right) = (2\pi\sigma_I^2)^{-\frac{N}{2}} \left|\mathbf{E}(\eta^2)\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma_I^2}\left(\mathbf{d} - \mathbf{H}\mathbf{a}\right)^T \mathbf{E}^{-1}(\eta^2)\left(\mathbf{d} - \mathbf{H}\mathbf{a}\right)\right]$$

$\eta^2\ (= \sigma_s^2/\sigma_I^2)$: relative weight of data sets ～ new hyperparameter

CASE2: **Joint inversion**

Expression of ABIC

▪Prior Information: $p\left(\mathbf{a};\rho_1^2,\rho_2^2\right)=(2\pi)^{-M/2}\left|\frac{\mathbf{G}_1}{\rho_1^2}+\frac{\mathbf{G}_2}{\rho_2^2}\right|^{1/2}\exp\left(-\frac{1}{2}\mathbf{a}^T\left(\frac{\mathbf{G}_1}{\rho_1^2}+\frac{\mathbf{G}_2}{\rho_2^2}\right)\mathbf{a}\right)$

(Fukahata *et al.*, 2003, 2004)

▪Bayes' Theorem

$$p\left(\mathbf{a};\sigma_I^2,\eta^2,\rho_1^2,\rho_2^2\,|\,\mathbf{d}\right)=c\,p\left(\mathbf{d}\,|\,\mathbf{a};\sigma_I^2,\eta^2\right)p\left(\mathbf{a};\rho_1^2,\rho_2^2\right)$$

▪Derivation of ABIC

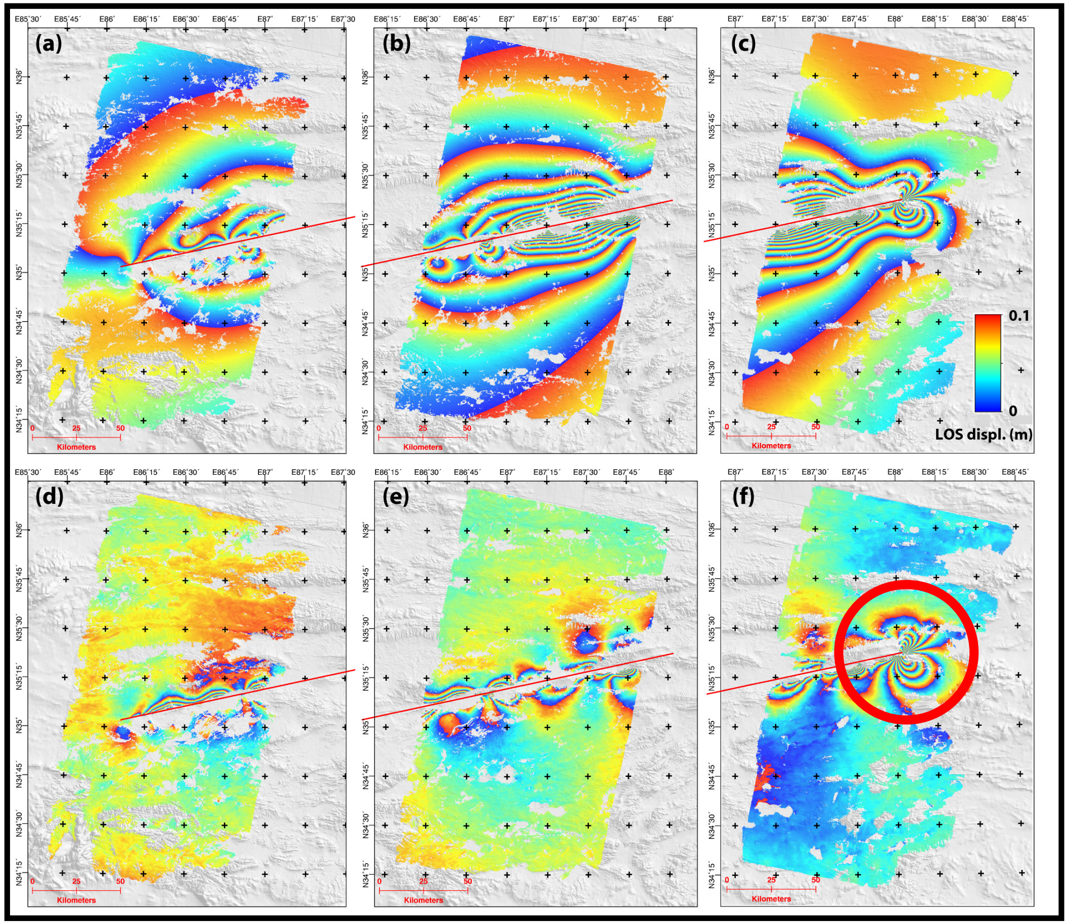
$$\mathrm{ABIC}\left(\alpha^2,\beta^2,\eta^2\right)=N\log s(\mathbf{a}^*)-\log\left|\alpha^2\mathbf{G}_1+\beta^2\mathbf{G}_2\right|$$
$$+\log\left|\mathbf{H}^T\mathbf{E}^{-1}(\eta^2)\mathbf{H}+\alpha^2\mathbf{G}_1+\beta^2\mathbf{G}_2\right|+\underline{\log\left|\mathbf{E}(\eta^2)\right|}+C$$

$$(\alpha^2=\sigma_I^2/\rho_1^2,\ \beta^2=\sigma_I^2/\rho_2^2)$$

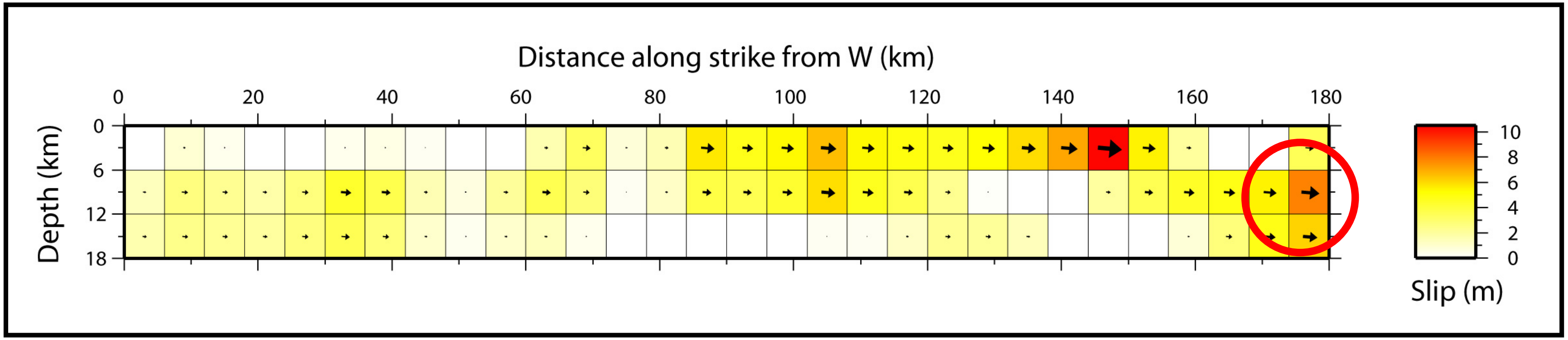
ABIC min → optimum values of α^2,β^2,η^2 → **a**

Results for only InSAR data

データとの比較

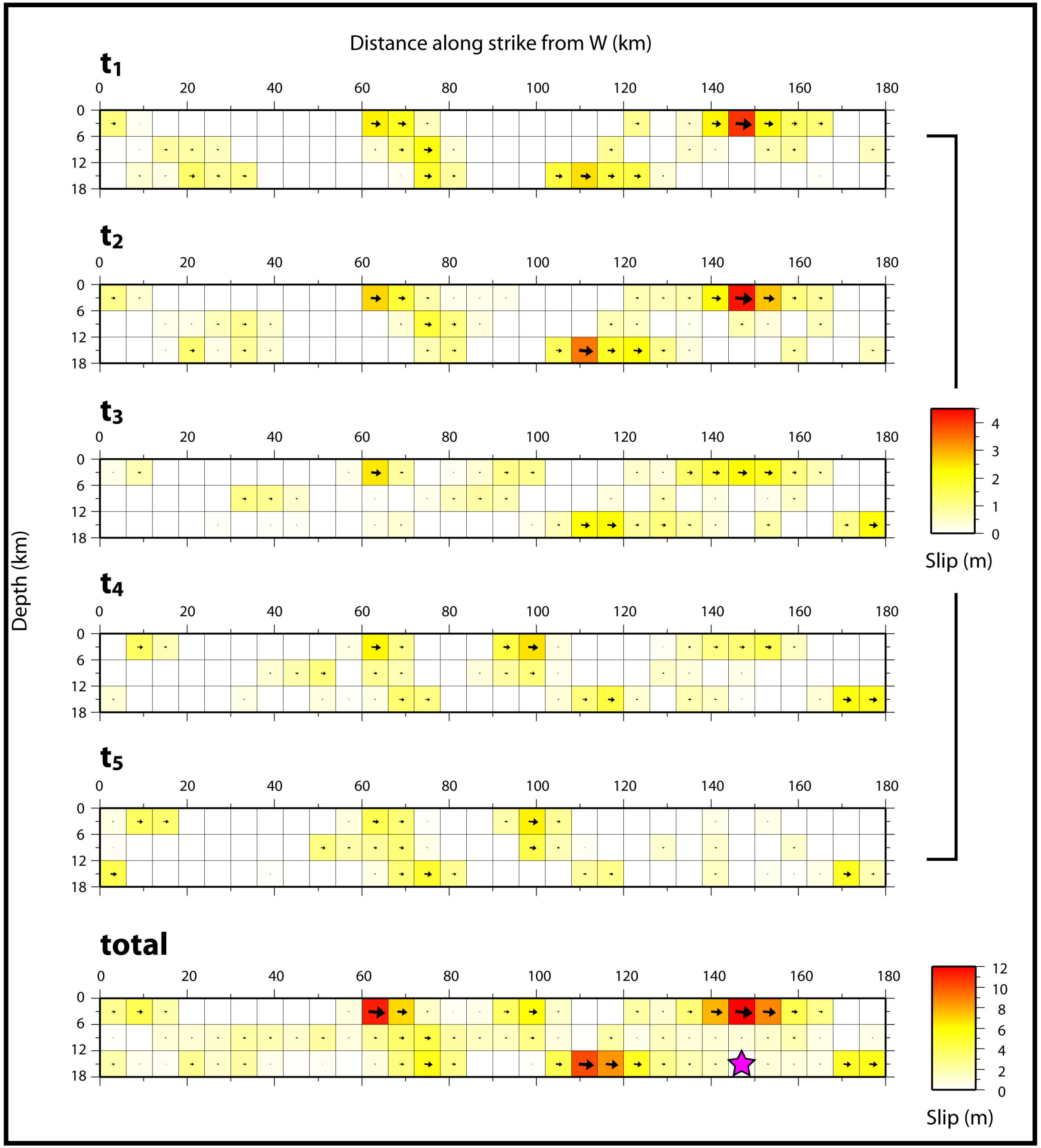


Slip Distribution

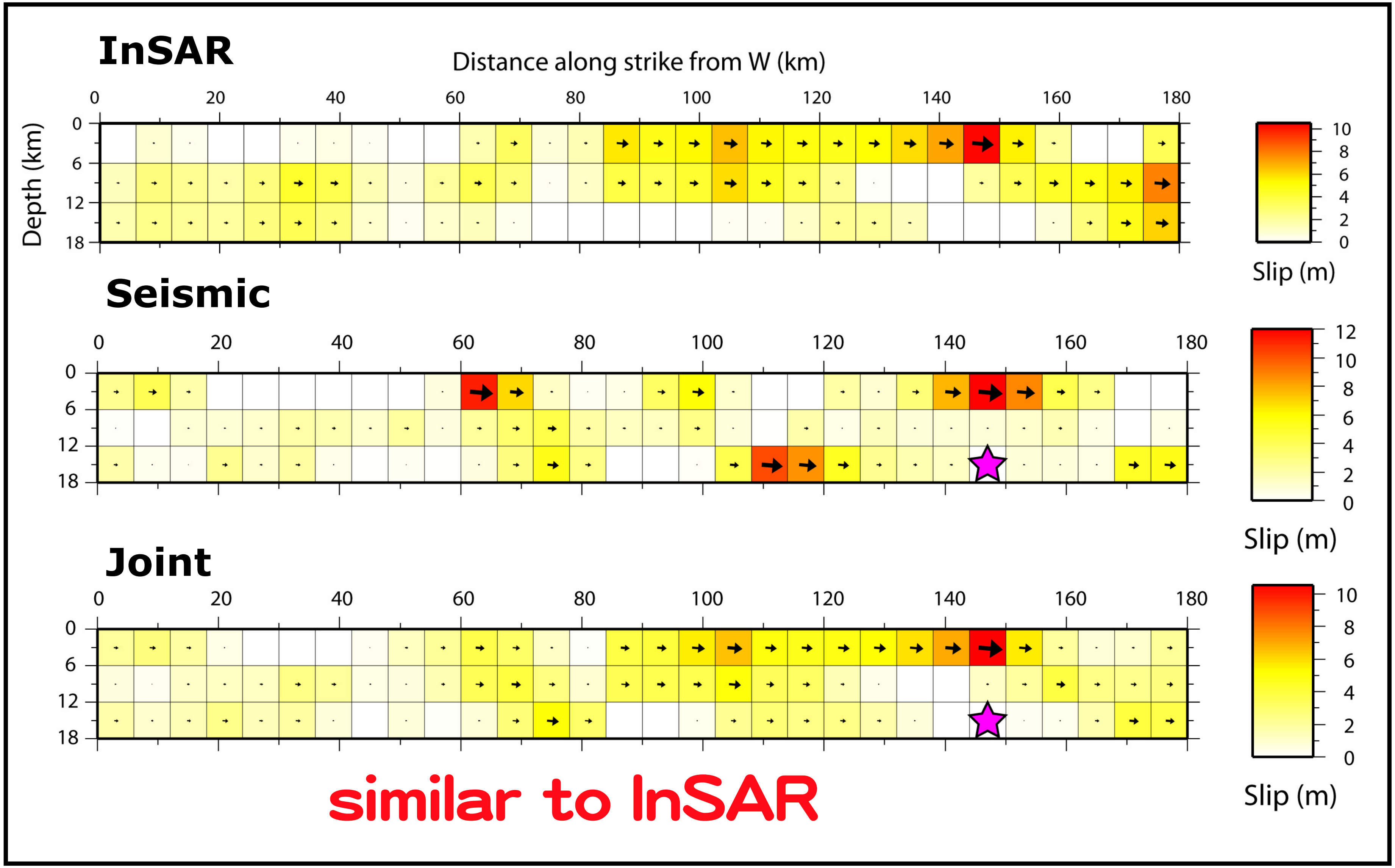


Funning, Fukahata, Yagi & Parsons (2014, GJI)

Results for Seismic Data



Result of Joint inversion



Funning, Fukahata, Yagi & Parsons (2014, GJI)

CASE3: **Weak Non-linear Inversion**

Expression of observation equation

Non-linear: $\mathbf{d} = f(\mathbf{a}, \mathbf{f}) + \mathbf{e}$ (f: non-linear function)

↓ *no approximation*

Pseudo linearization: $\mathbf{d} = \mathbf{H}(\mathbf{f})\mathbf{a} + \mathbf{e}$ (**H**: matrix)

The parameters responsible for non-linearity is separated from model parameters **a** *that express slip distribution*

where $\mathbf{f}$: fault parameters (dip, strike, location)

$$u(x,z;\delta) = \sum_{k=1}^{K}\sum_{l=1}^{L} a_{kl} X_k(x) Z_l(z)$$: slip distribution

a_{kl} : model parameters

$X_k(x), Z_l(z)$: basis functions

Determination of the non-linear parameters **f** with ABIC

- Residual sum of squares with prior (smoothness constraint)

$$s(\mathbf{a};\alpha^2,\mathbf{f}) = (\mathbf{d}-\mathbf{H}(\mathbf{f})\mathbf{a})^T\mathbf{E}^{-1}(\mathbf{d}-\mathbf{H}(\mathbf{f})\mathbf{a}) + \alpha^2\mathbf{a}^T\mathbf{G}\mathbf{a}$$

cf. $s(\mathbf{a};\alpha^2) = (\mathbf{d}-\mathbf{H}\mathbf{a})^T\mathbf{E}^{-1}(\mathbf{d}-\mathbf{H}\mathbf{a}) + \underline{\alpha^2\mathbf{a}^T\mathbf{G}\mathbf{a}}$: linear case

Non-linear term

α^2 : weight between observation and prior

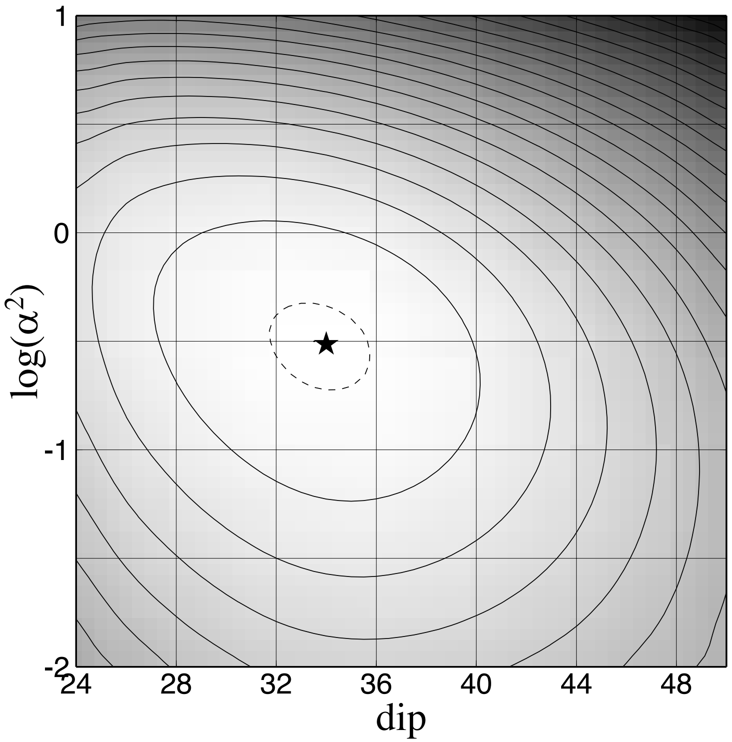
↑

ABIC min (Yabuki & Matsu'ura, 1992)

In the same way, **f** ⟵ ABIC min

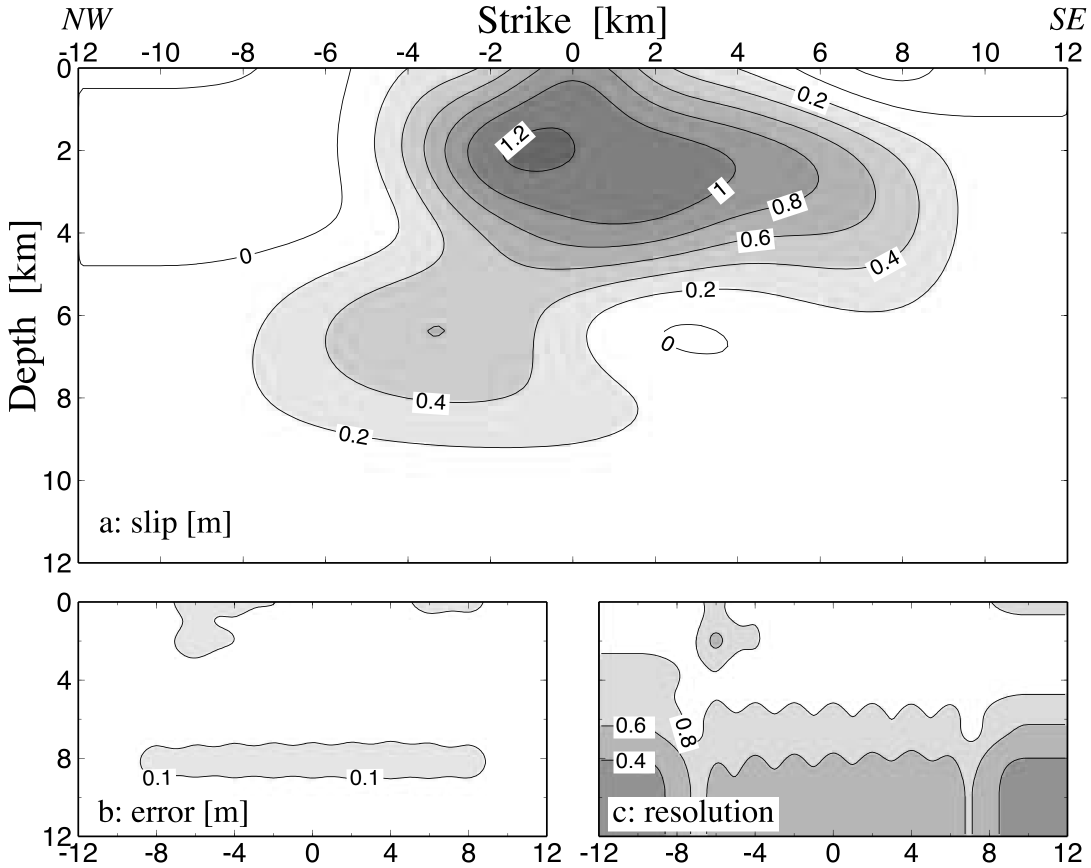
Fukahata & Wright (2008, GJI)

最適傾斜角の推定



実線は20毎のABICコンター.
点線は最小値から2のコンターで
誤差範囲に相当.

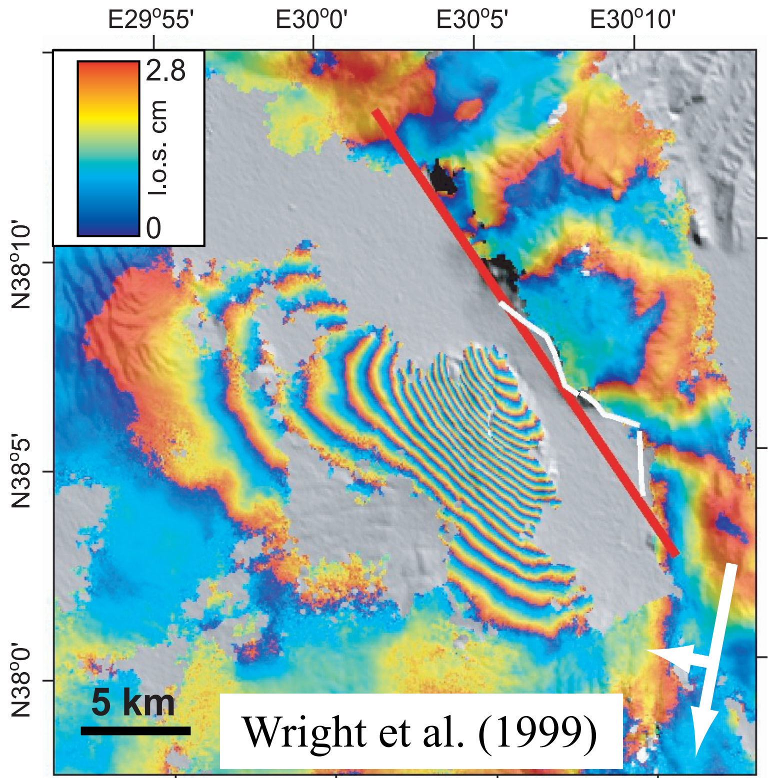
誤差付きの滑り分布推定



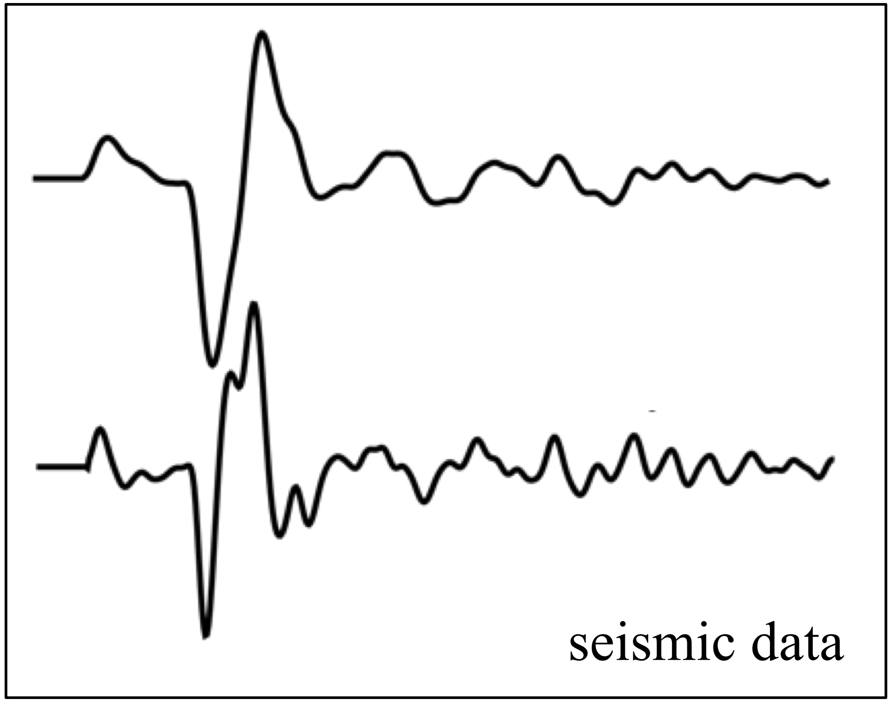
Fukahata & Wright (2008, GJI)

4. Covariance components due to observation error

We have nominally continuous observed data

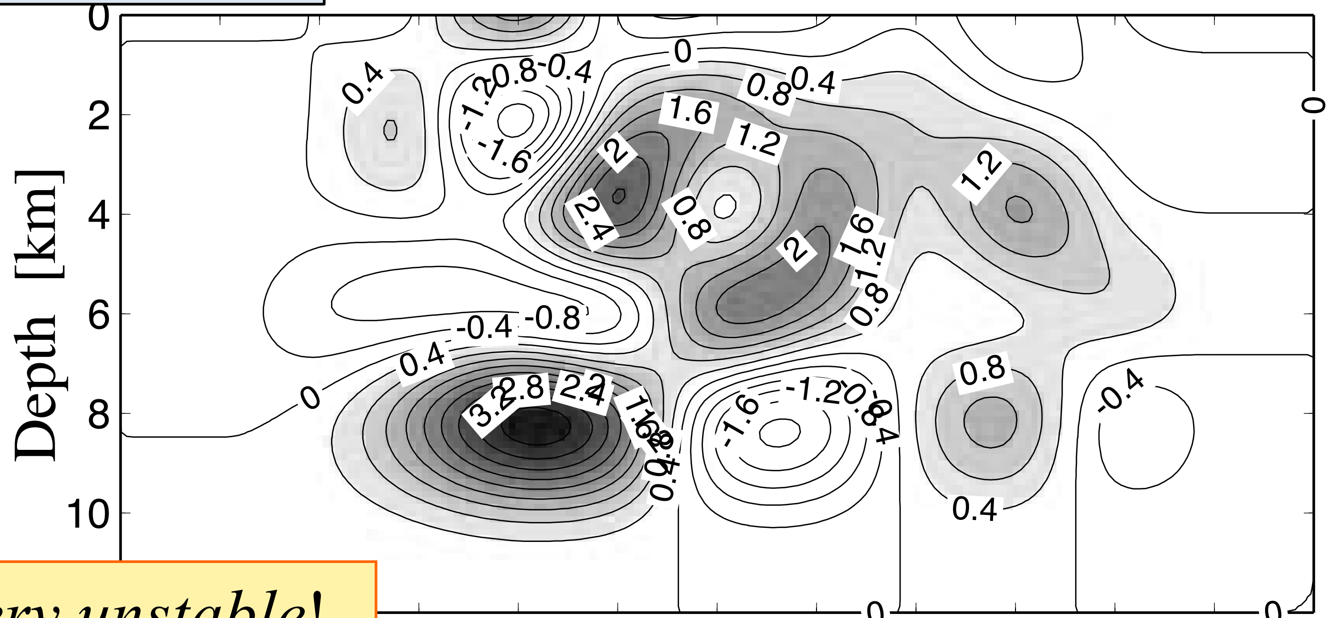


InSAR: one-million data



How should we sample and invert such data?

An inverted result



very unstable!

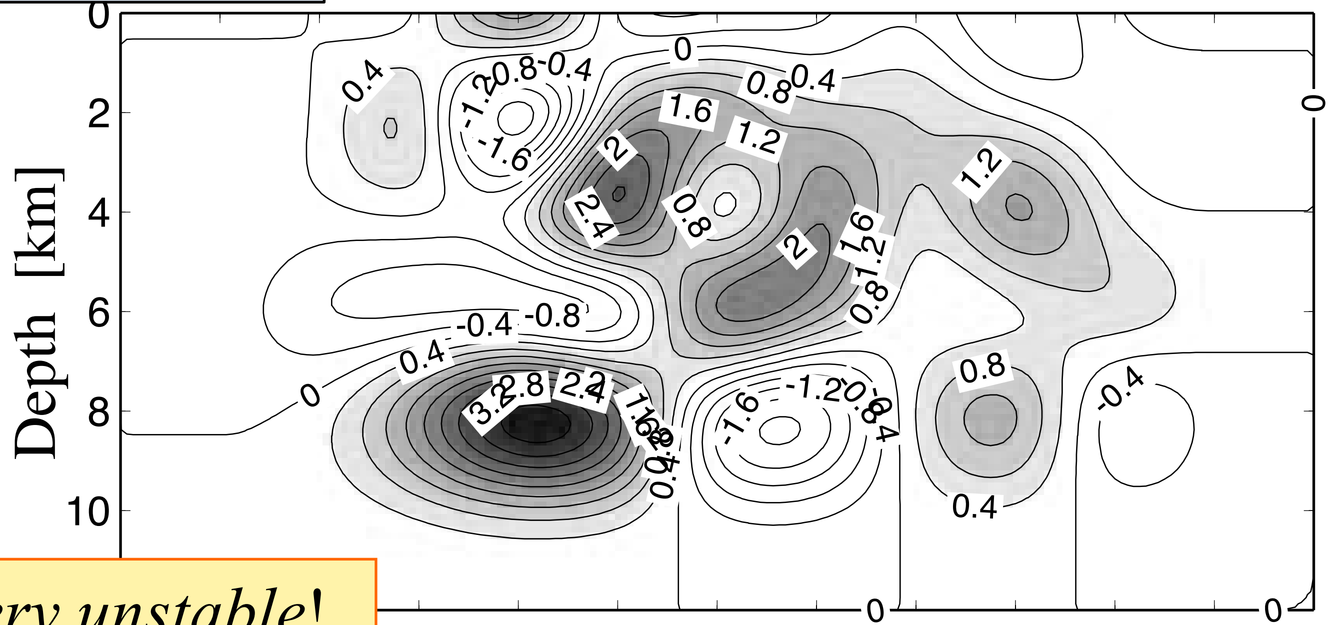
Slip distribution of Dinar earthquake

I reviewed the computation program,

tried various settings, etc,

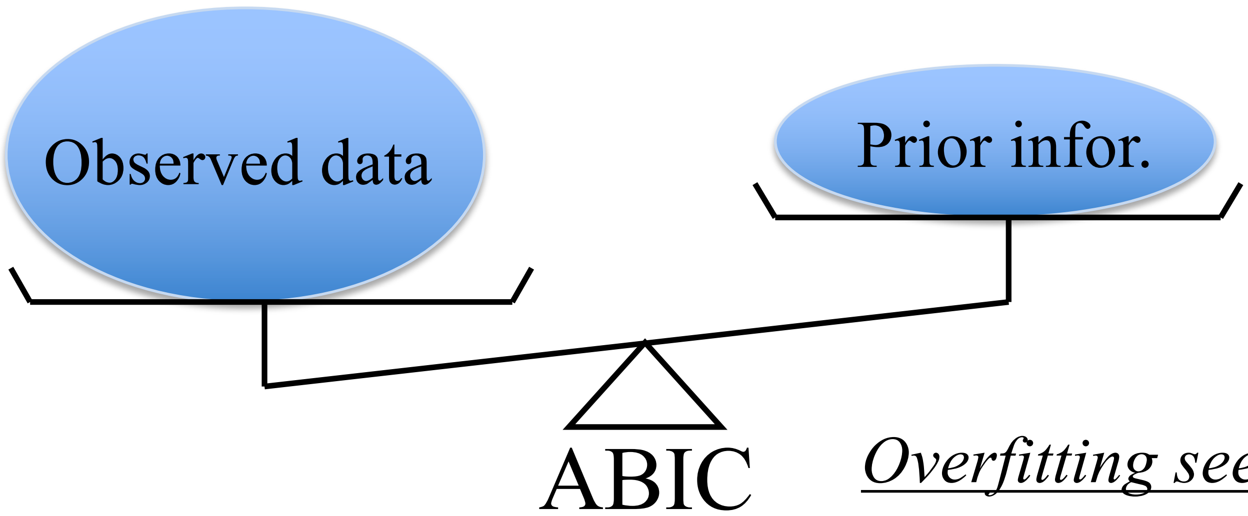
but the situation didn’t improve.

An inverted result

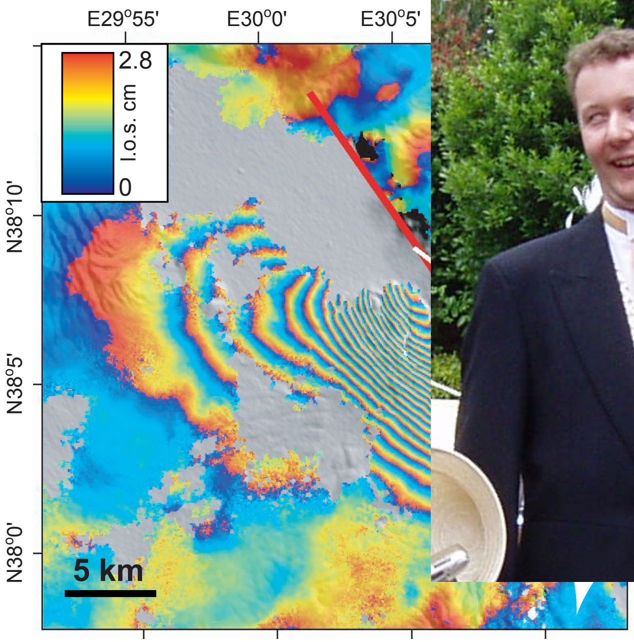


very unstable!

Slip distribution of Dinar earthquake



Overfitting seemed to occur.



At a coffee time, July 2005

me "I wonder 100 m might be too short for the correlation distance."

Tim "Yes, it's about ten kilometer"

me "Oh, really?"

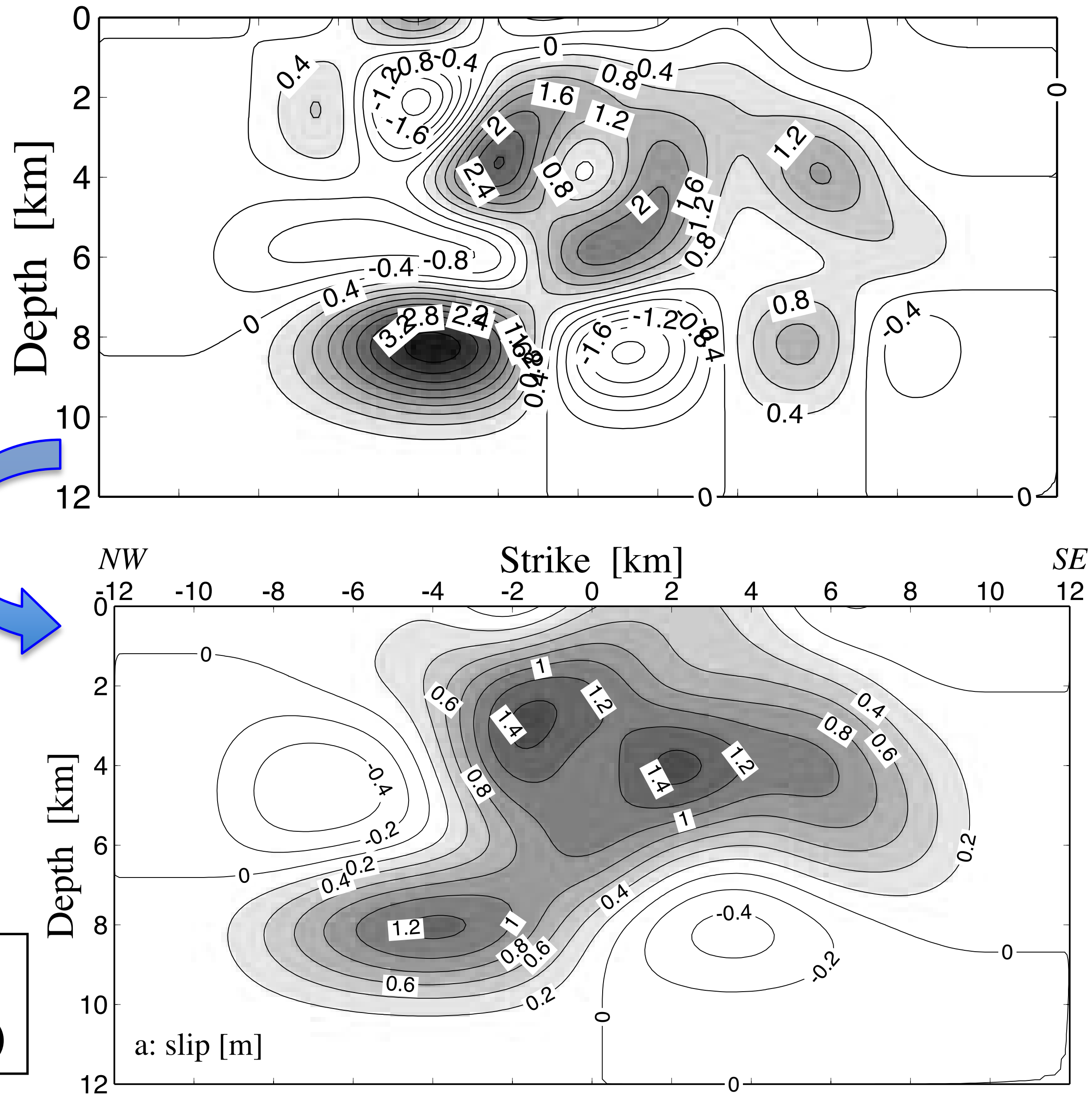
Observation Eq.: $\mathbf{d} = \mathbf{Ha} + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{E})$

$$s(\mathbf{a}; \alpha^2) = \underbrace{(\mathbf{d} - \mathbf{Ha})^T \mathbf{E}^{-1} (\mathbf{d} - \mathbf{Ha})}_{\text{square of residual}} + \underbrace{\alpha^2 \mathbf{a}^T \mathbf{Ga}}_{\text{smoothness}}$$

$$\mathbf{E} = \mathbf{I} \Rightarrow E_{ij} = \exp\left(-r_{ij}/s\right)$$

r_{ij} : distance between data i and j

s : typical correlation length (~10km)



Introduction of covariance of observation errors

Fukahata & Wright (2008)

Important lesson: Bad modelling leads to a bad result.

A bad result strongly suggests bad modelling.

Another important lesson

It was possible to manually adjust the hyperparameter to obtain a good-looking result. But, if I did so, probably I didn't realize the importance of covariance components.

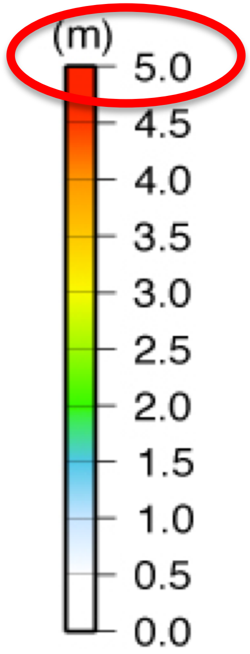
↓

The hyperparameters should be determined statistically (objectively).

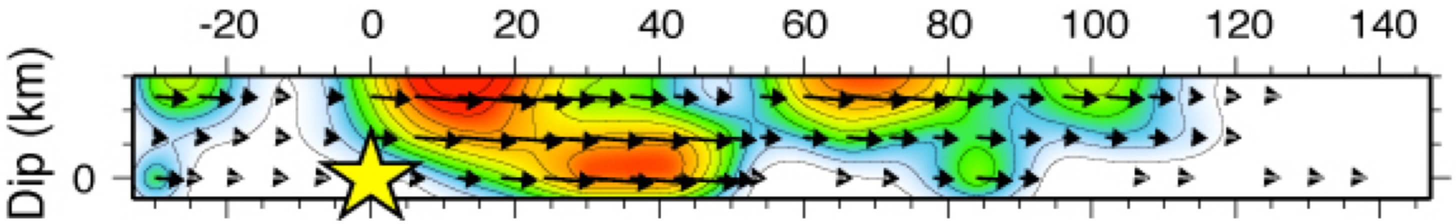
5. Covariance components due to modeling error

For the case of waveform inversion

Tibet, Manyi earthquake (Mw 7.6) (by Yagi on behalf of G. Funning)

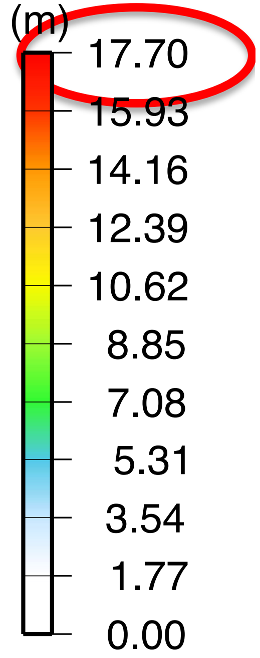


<Before> ***reasonable***

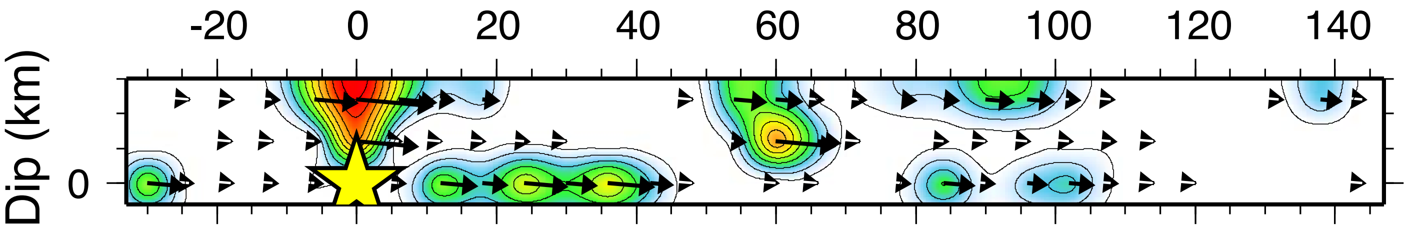


Yagi said,

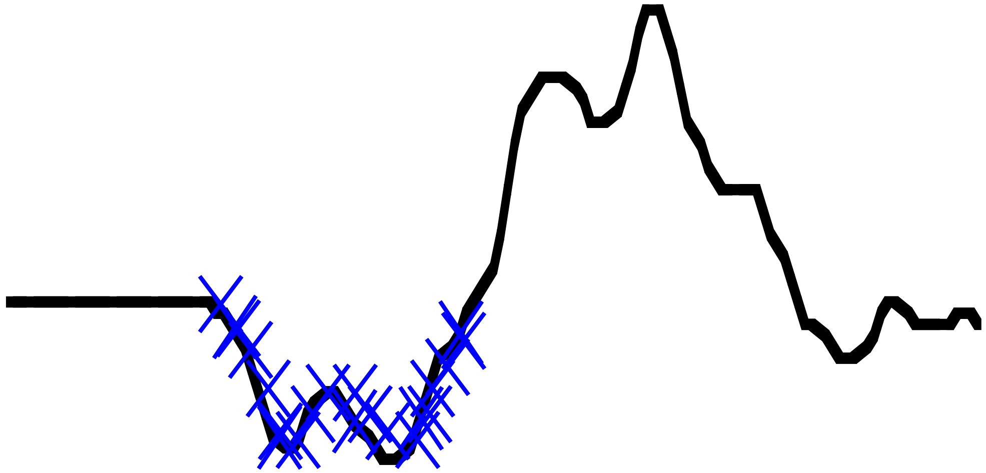
- improved P-wave picks
- etc, etc,
- ***made a sampling rate higher***



<After> ***extremely rough***



Waveform data are basically accurate, so observation errors are small.



But if we densely sample, data include common error.

Mathematically, covariance matrix E becomes

$$\mathsf{E} = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma^2 & * & \cdots & 0 \\ * & \sigma^2 & * & * \\ \vdots & * & \ddots & * \\ 0 & * & * & \sigma^2 \end{pmatrix}$$

We introduced **modeling error** (discretization error), which leads to covariance components.

$$u(\mathbf{x},t) = \sum_k \sum_l a_{kl} X_k(\mathbf{x}) T_l(t) + \underline{\delta u(\mathbf{x},t)}$$

$X_k(\mathbf{x}), T_l(t)$: Basis functions

(Yagi & Fukahata, 2008)

Introduction of covariance components due to modeling error (discretization error)

(Yagi & Fukahata, 2008)

discretization error: $$u(\tau) = \sum_{k=1}^{K} a_k T_k(\tau) + \delta u(\tau)$$

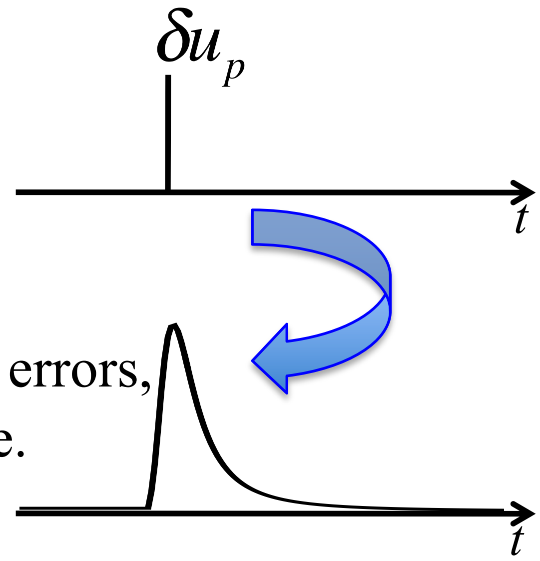
$T_k(t)$: basis function

When parameterizing the problem, discretization error inevitably emerges

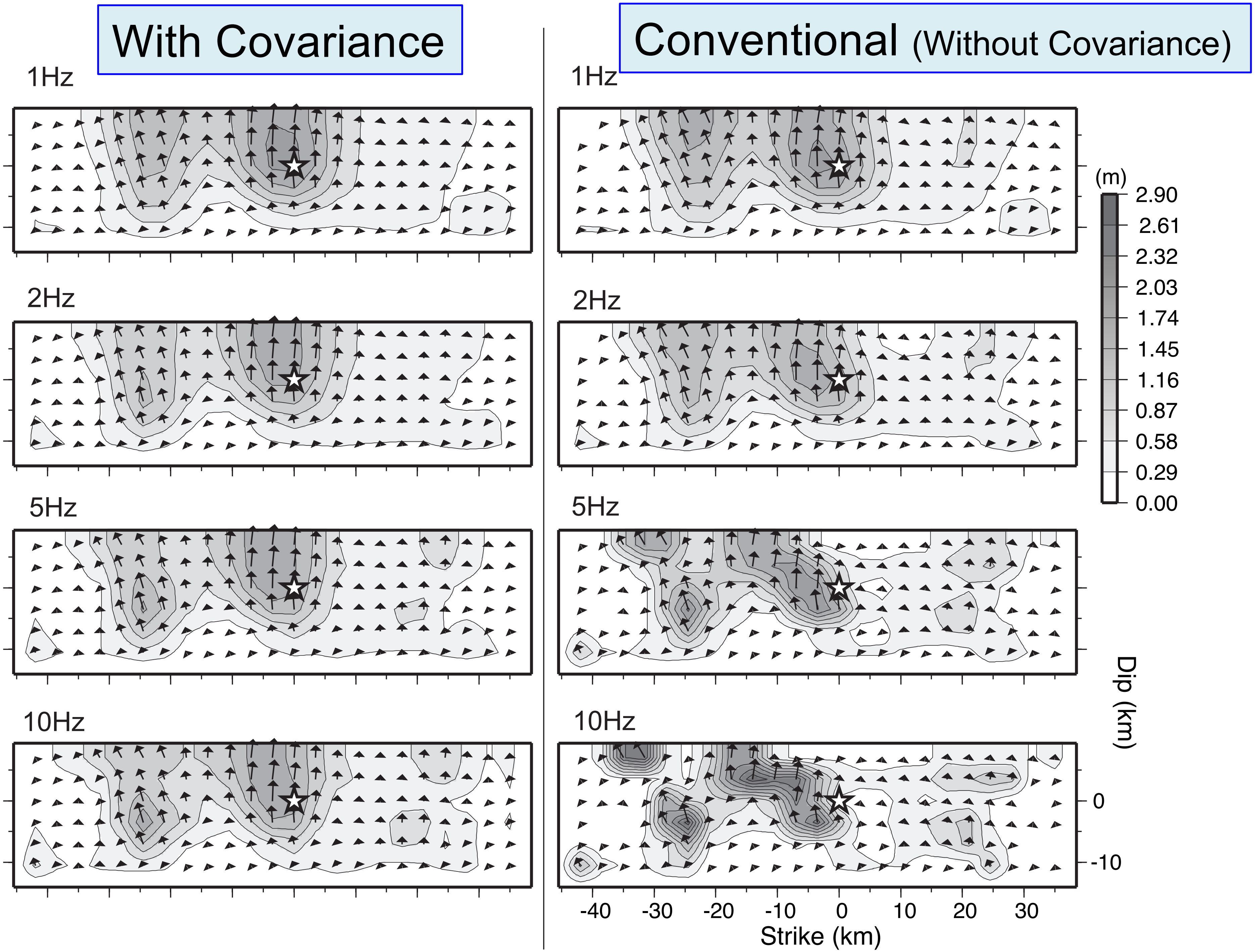
Relation between data and model: $$d_i(t) = \int_s G_i(t;\tau)u(\tau)d\tau$$

Expression of the error in the observation eq.:

$$e_i^{discre}(t) = \int_s G_i(t;\tau)\delta u(\tau)d\tau$$

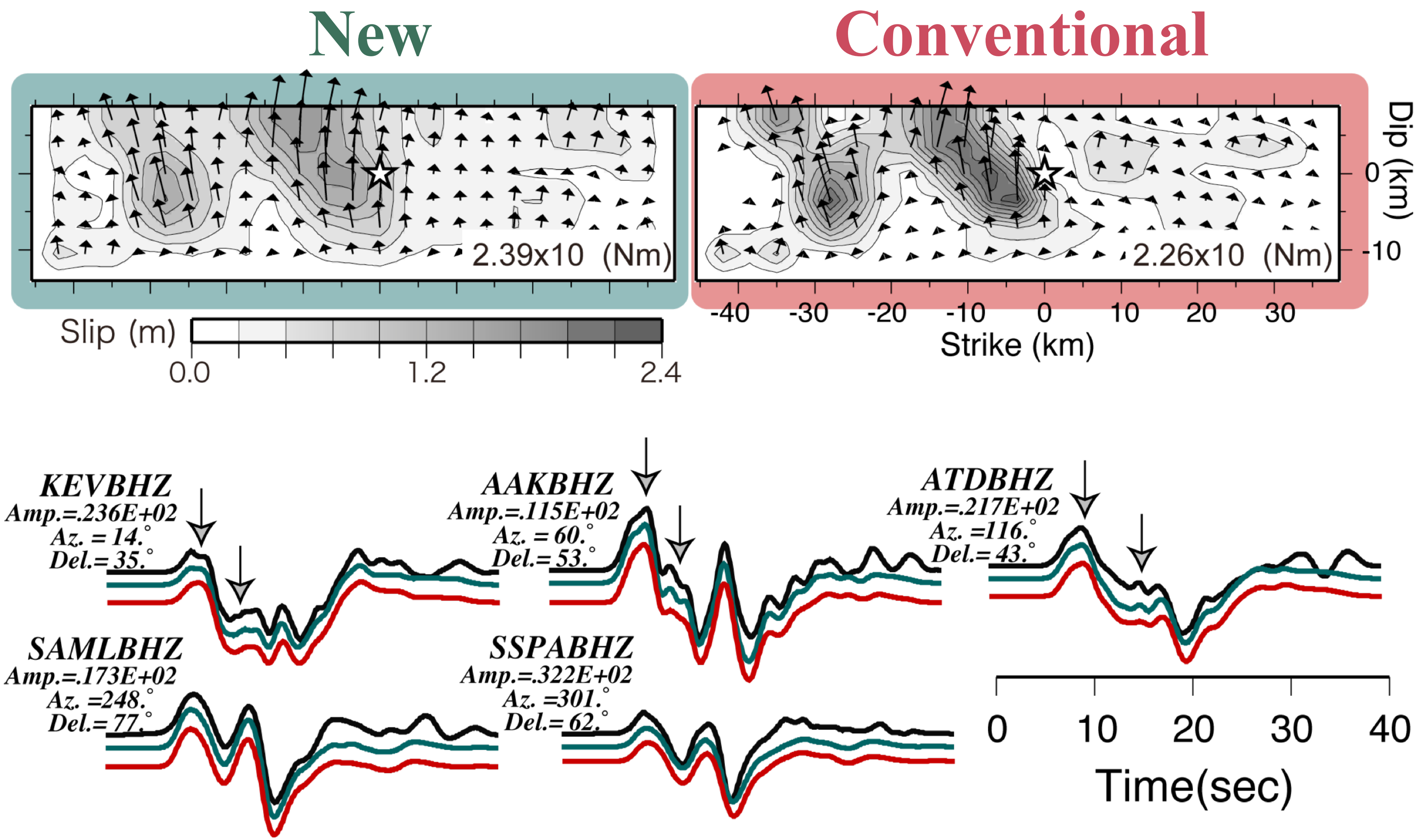


Following the law of propagation of errors, covariance components emerge.



(Yagi & Fukahata, 2008)

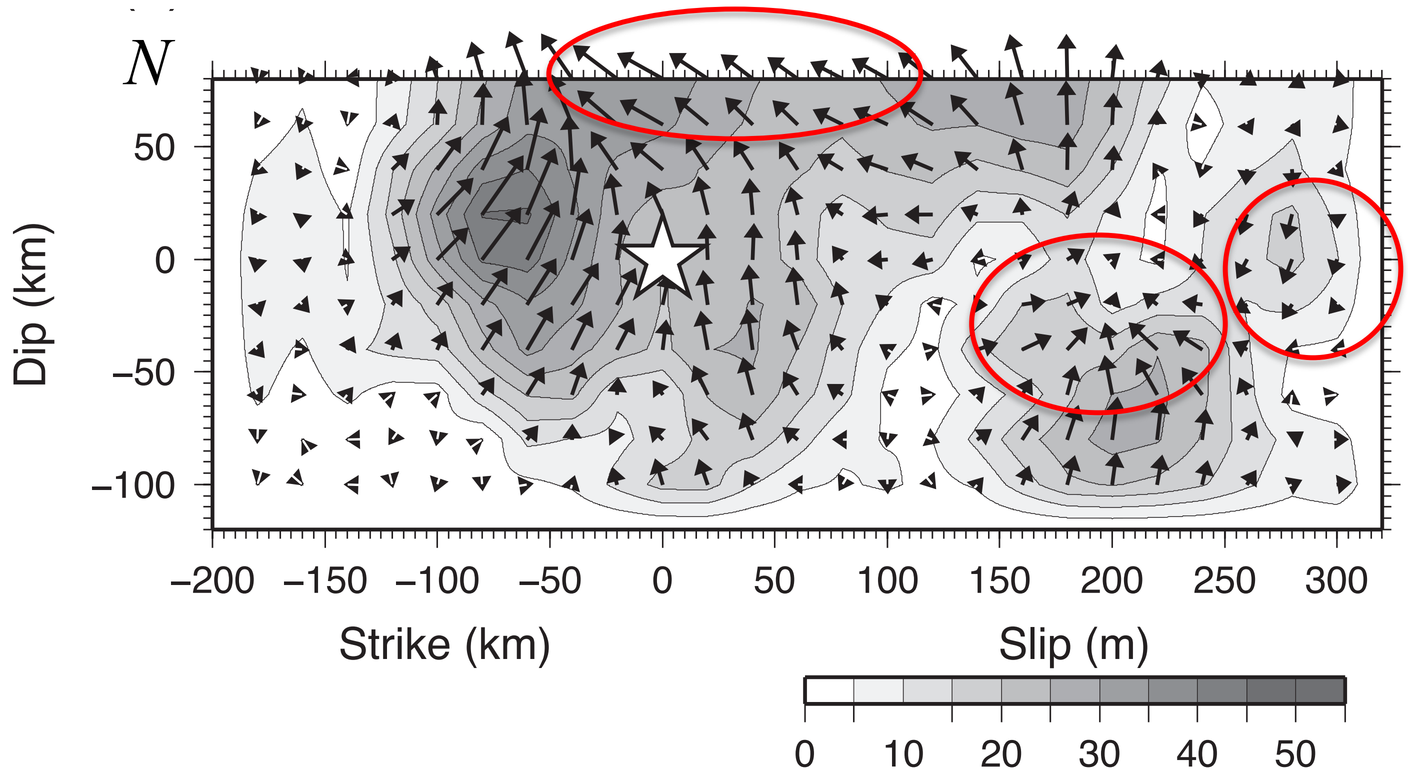
Reproduction of high frequency components



Note that the residual mean square is *less* in the conventional

5-2. Covariance due to uncertainty of Green's function

Another Example of unrealistic result



Tohoku-oki (M9.0), 2011

Fukahata, Yagi & Mitsui (2012)

Large negative & strike slips

which also tells us something is wrong in the inversion scheme

(We should be very careful to apply non-negative condition)

<Non-negative Condition>

Non-negative condition is commonly considered to be ***physically reasonable***. However, the non-negative condition always leads to biased estimates, i.e.,

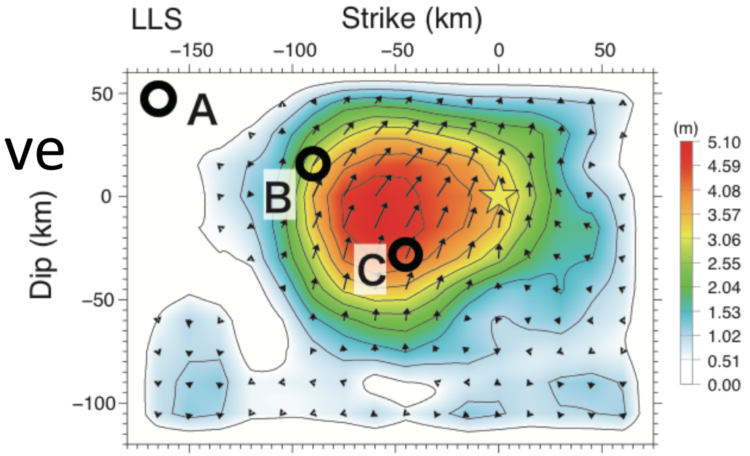
$$E\left(u(\mathbf{x})\right)=0 \quad \xrightarrow{\text{non-negative}} \quad E\left(u(\mathbf{x})\right)>0$$

E: expectation
u: slip
$\mathbf{x}$: a far distance from the source

"Unbiased" is one of the most important criterion in statistical inference.

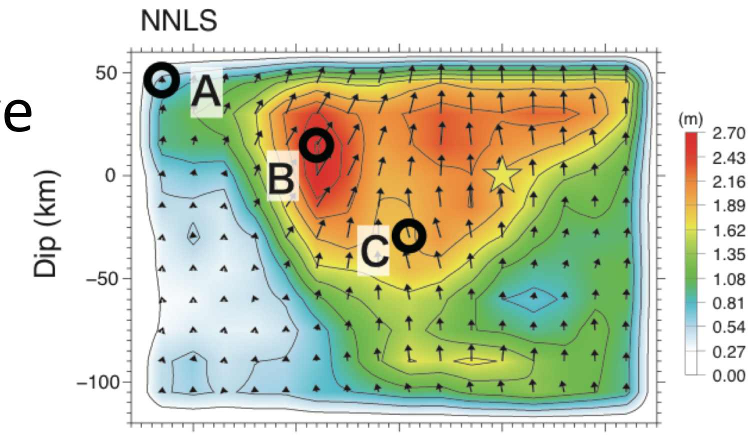
[Max slip]

without Non-Negative Condition (LLS)



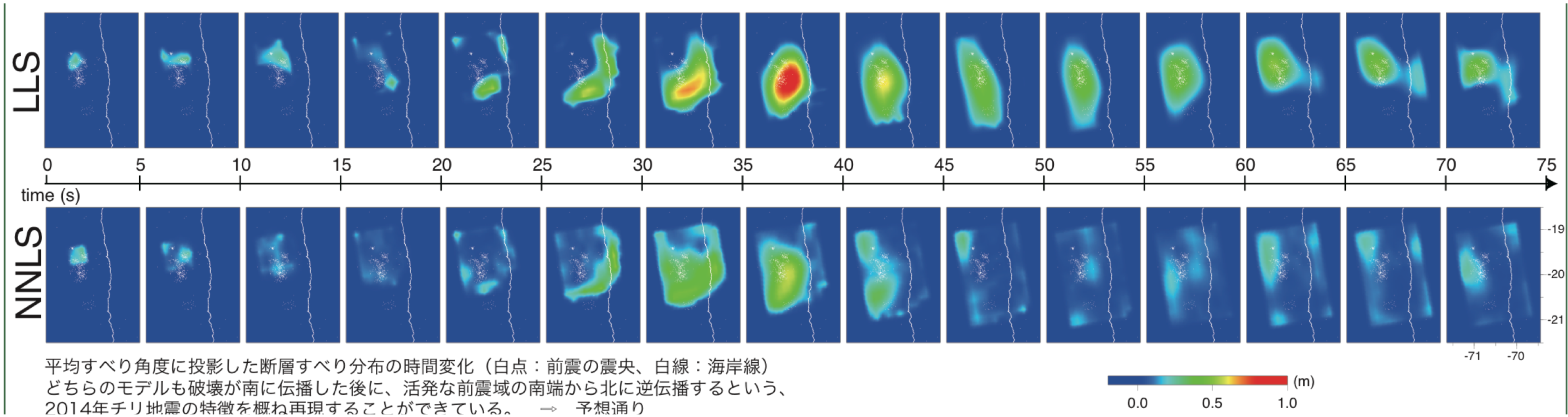
5.0 m

with Non-Negative Condition (NLLS)



2.5 m

Yagi & Fukahata (2014, SSJ)



平均すべり角度に投影した断層すべり分布の時間変化（白点：前震の震央、白線：海岸線）
どちらのモデルも破壊が南に伝播した後に、活発な前震域の南端から北に逆伝播するという、
2014年チリ地震の特徴を概ね再現することができている。 ⇒ 予想通り

Introduction of uncertainty of Green's function

(Yagi & Fukahata, 2011; GJI)

$$\mathbf{e} = \mathbf{e}^{obs} + \mathbf{e}^{model}$$

Observation equation : $\mathbf{d} = \mathbf{G}\mathbf{a} + \mathbf{e}^{obs}$

Introduction of uncertainty of Green's function

$$\mathbf{d} = (\mathbf{G} + \delta\mathbf{G})\mathbf{a} + \mathbf{e}^{obs}$$

$$\mathbf{d} = \mathbf{G}\mathbf{a} + (\underline{\delta\mathbf{G}\mathbf{a} + \mathbf{e}^{obs}})$$

New error term

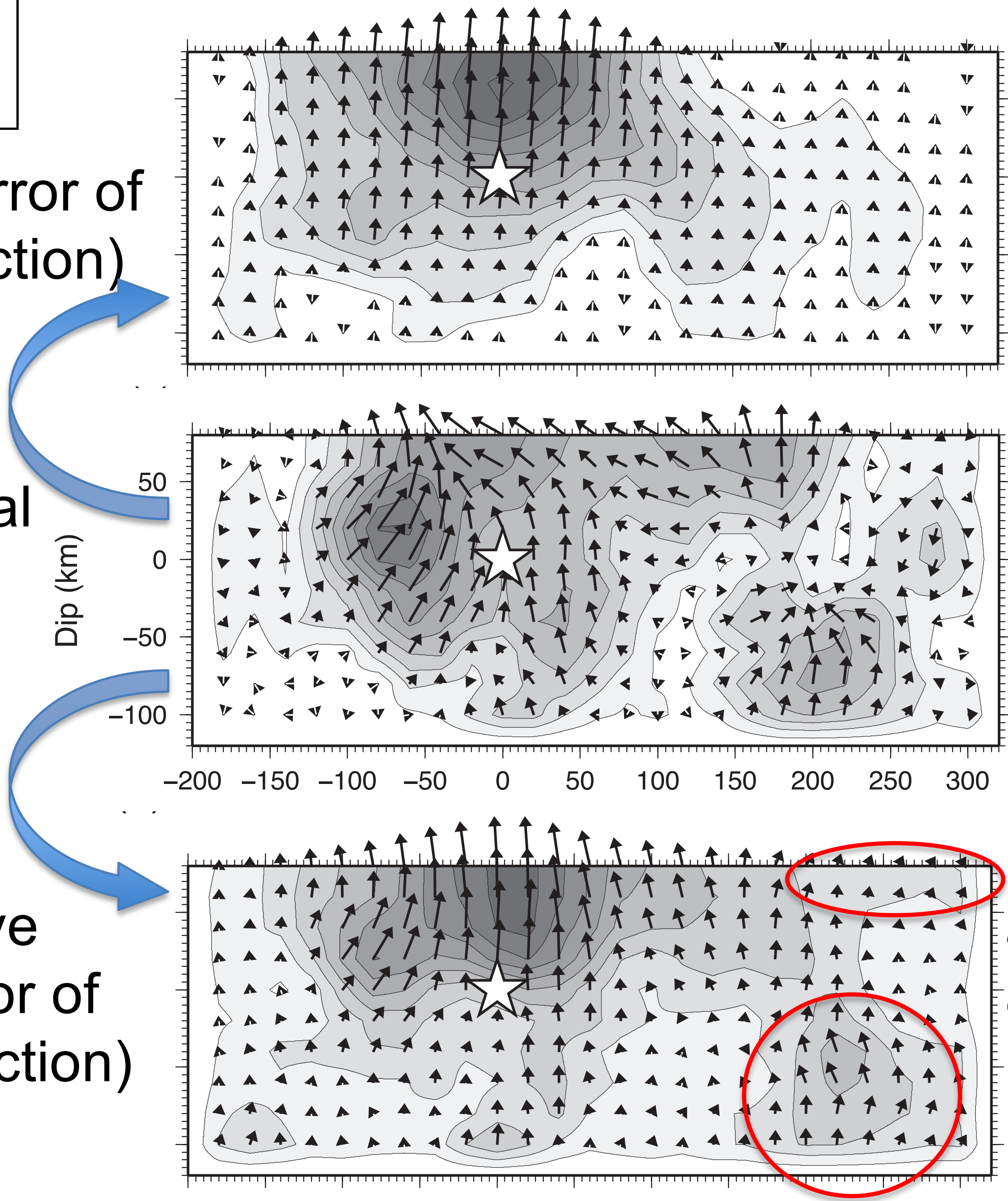
$\delta\mathbf{G}$ is assumed to be Gaussian for simplicity.

Slip Distribution of Tohoku eq.

New (with error of Green's function)

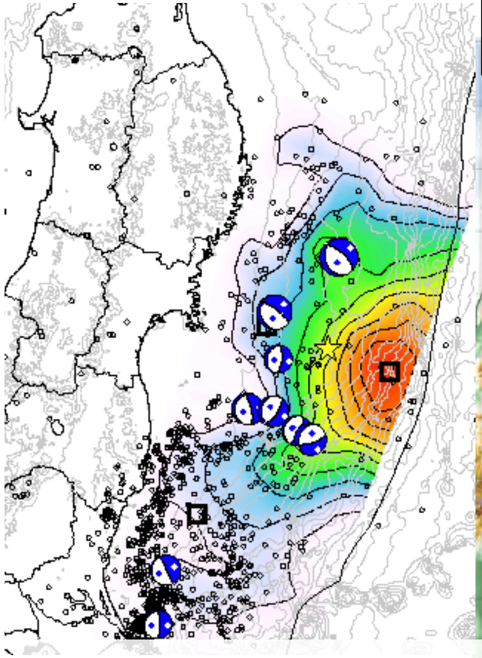
Conventional

Non-negative (without error of Green's function)

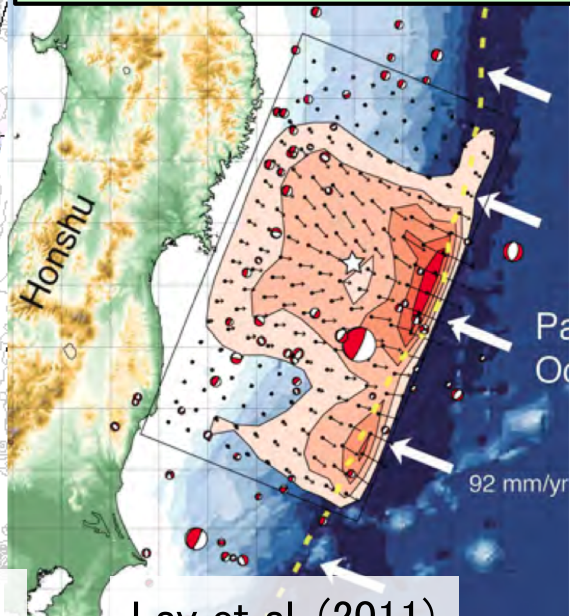


likely to be artificial

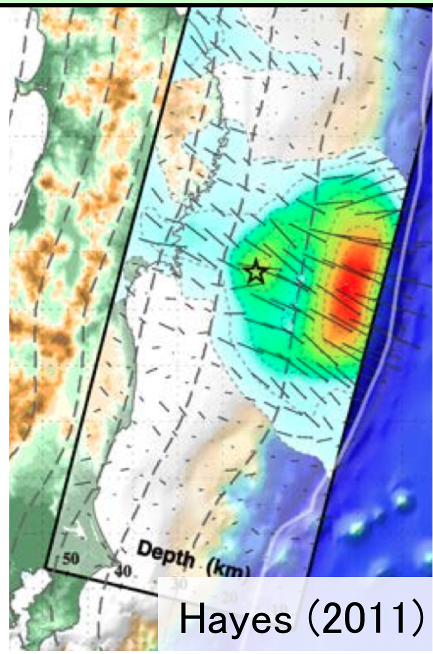
Inversion results of the Tohoku-oki earthquake



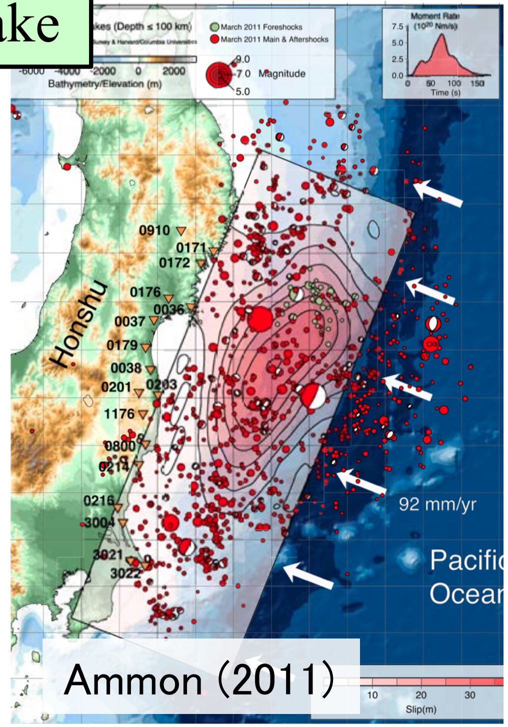
Yagi & Fukahata (2011)



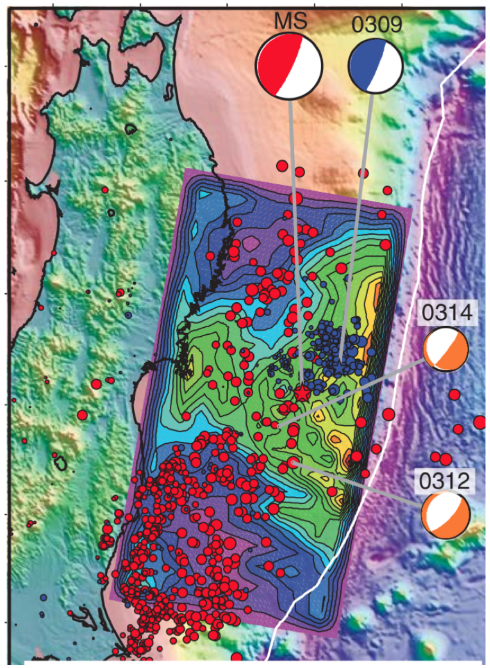
Lay et al. (2011)



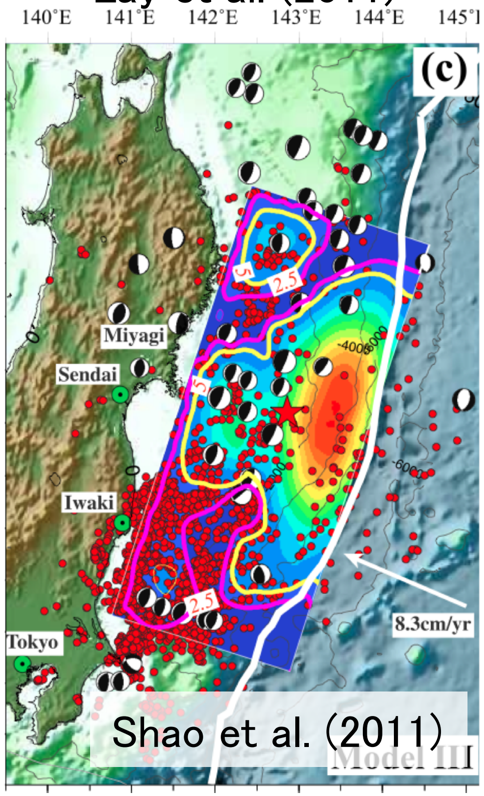
Hayes (2011)



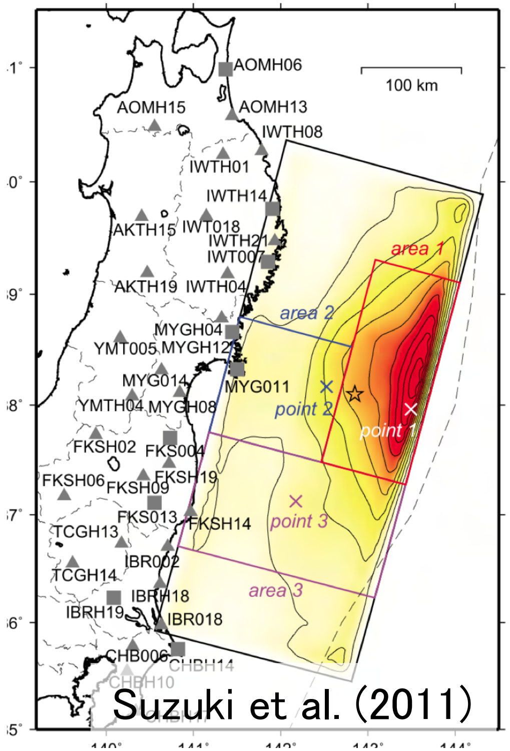
Ammon (2011)



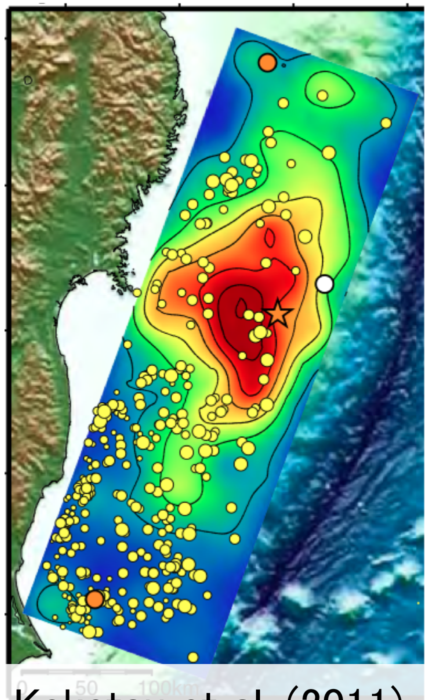
Ide et al. (2011)



Shao et al. (2011)



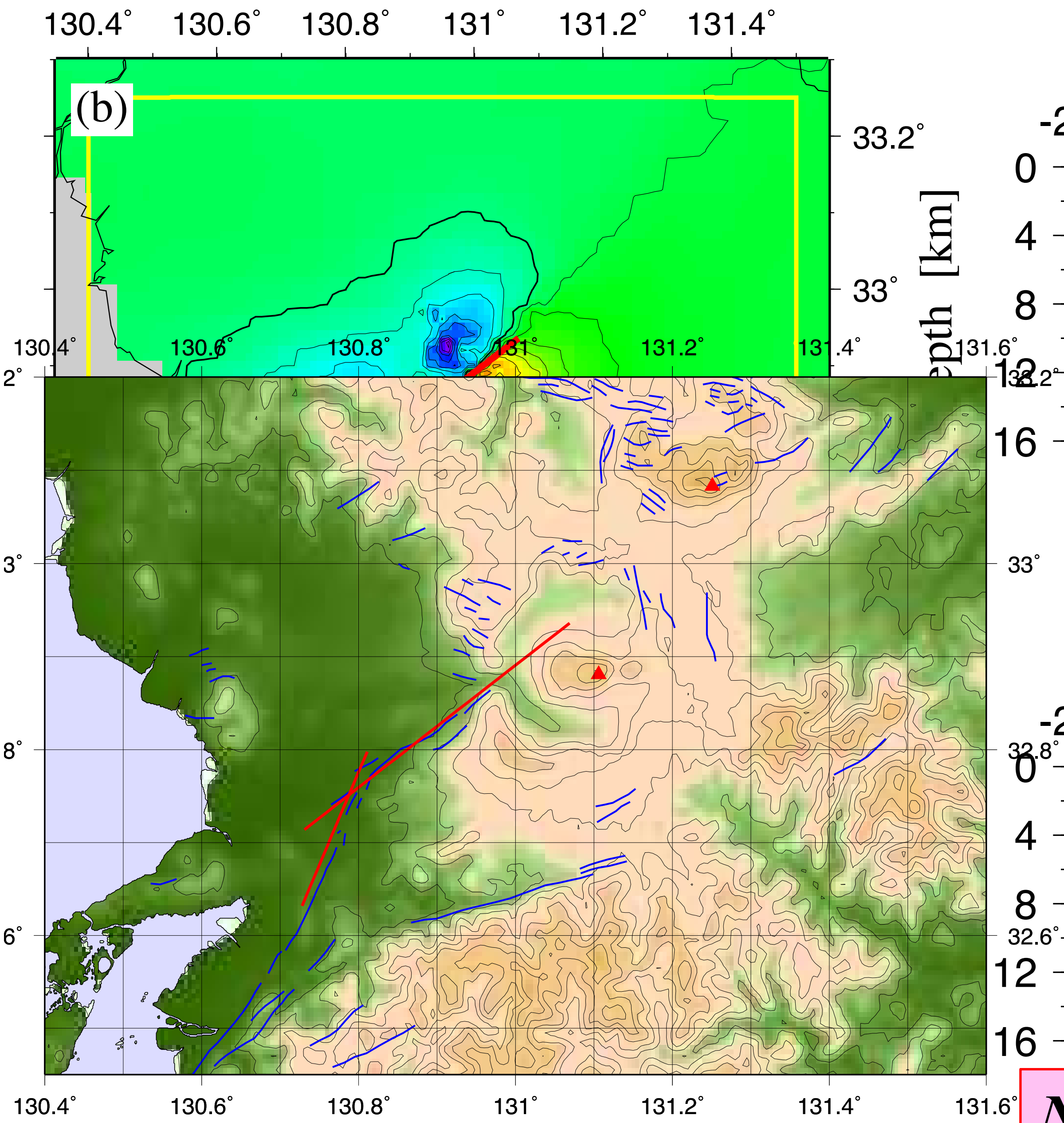
Suzuki et al. (2011)



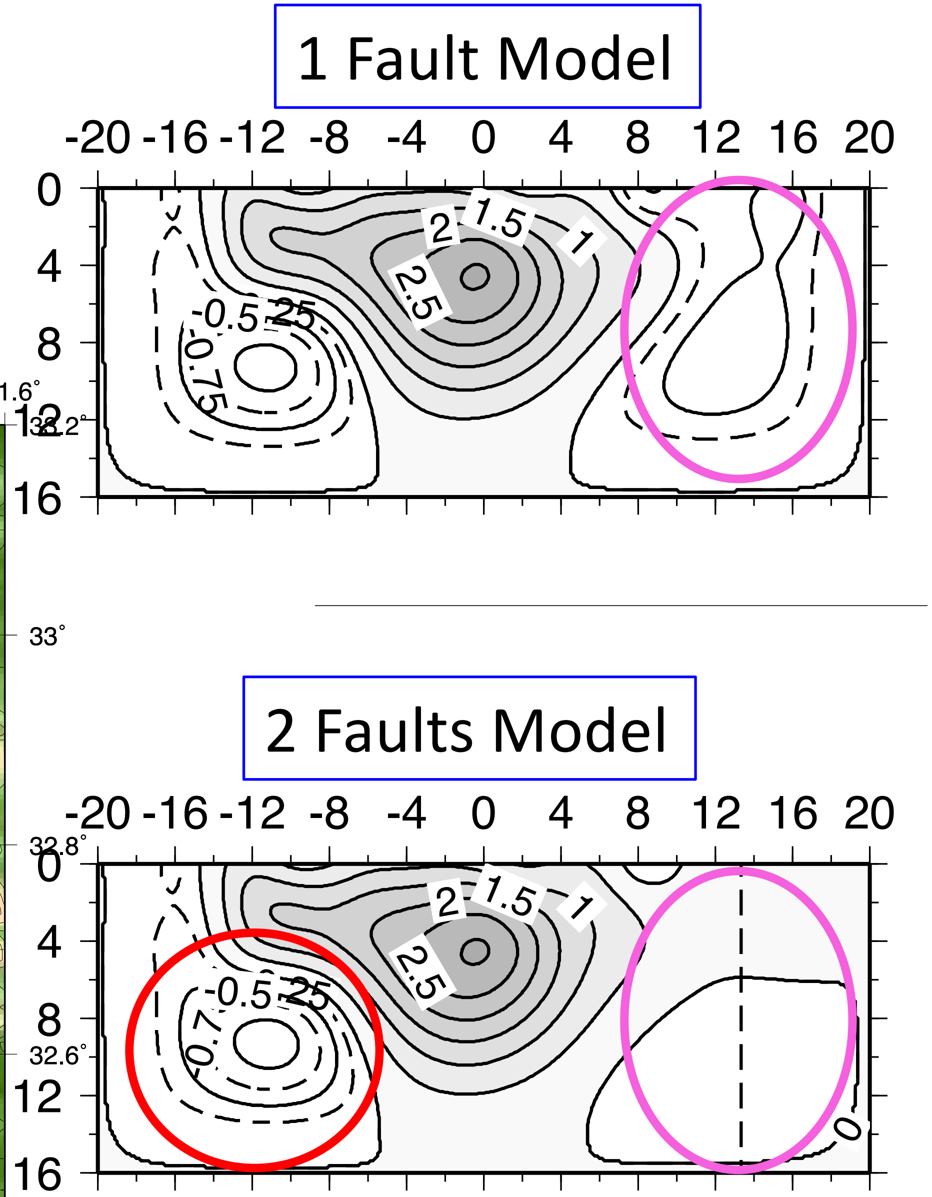
Koketsu et al. (2011)

Error due to setting of a fault model

2016 Kumamoto earthquake



Fukahata & Hashimoto (2016, EPS)



Negative slip diminishes!

Summary

In order to obtain apparently good looking results, we shouldn't adjust the smoothing parameter and/or apply non-negative condition.

Physically unrealistic results strongly suggest that something is wrong in the setting of the inversion analysis.

By seeking the reason for unrealistic results, we can develop a better inversion method

"*Bad modelling leads to a bad result.*
A bad result suggests bad modelling."